Three-dimensional oscillator and Coulomb systems reduced from Kähler spaces

2004 J. Phys. A: Math. Gen. 37 2791

(http://iopscience.iop.org/0305-4470/37/7/020)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2791–2801 PII: S0305-4470(04)70935-X

Three-dimensional oscillator and Coulomb systems reduced from Kähler spaces

Armen Nersessian[1,2] **and Armen Yeranyan**[1]

[1] Yerevan State University, Alex Manoogian Street, 1, Yerevan, 375025, Armenia
[2] Yerevan Physics Institute, Alikhanian Brothers Street, 2, Yerevan, 375036, Armenia

Received 23 October 2003
Published 4 February 2004
Online at stacks.iop.org/JPhysA/37/2791 (DOI: 10.1088/0305-4470/37/7/020)

Abstract
We define the oscillator and Coulomb systems on four-dimensional spaces with $U(2)$-invariant Kähler metric and perform their Hamiltonian reduction to the three-dimensional oscillator and Coulomb systems specified by the presence of Dirac monopoles. We find the Kähler spaces with conic singularity, where the oscillator and Coulomb systems on three-dimensional sphere and two-sheet hyperboloid originate. Then we construct the superintegrable oscillator system on three-dimensional sphere and hyperboloid, coupled to a monopole, and find their four-dimensional origins. In the latter case the metric of configuration space is a non-Kähler one. Finally, we extend these results to the family of Kähler spaces with conic singularities.

PACS numbers: 03.65.−w, 11.30.Pb

To the memory of Professor Valery Ter-Antonyan

1. Introduction

The oscillator and Coulomb systems play a distinguished role in theoretical and mathematical physics due to their overcomplete symmetry group. The wide number of hidden symmetries provides these systems with unique properties, e.g., closed classical trajectories, the degenerate quantum–mechanical energy spectrum, the separability of variables in a few coordinate systems. The overcomplete symmetry allows us to preserve their exact solvability even after some deformation of the potential breaking the initial symmetry, or, at least, to simplify the perturbative calculations. The reduction of these systems to low dimensions allows one to construct new integrable systems with hidden symmetries [1].

There exist non-trivial generalizations of the oscillator and Coulomb systems on the sphere and the two-sheet hyperboloid (pseudosphere) [2] given by the potentials

$$V_{\text{Coulomb}} = -\frac{\gamma}{r_0}\frac{x_{d+1}}{|\mathbf{x}|} \qquad V_{\text{osc}} = \frac{\omega^2 r_0^2}{2}\frac{\mathbf{x}^2}{x_{d+1}^2}. \tag{1.1}$$

Here $\mathbf{x}, x_{d+1}$ are the (pseudo)Euclidean coordinates of the ambient space $\mathbb{R}^{d+1}(\mathbb{R}^{d.1})$: $\epsilon\mathbf{x}^2 + x_{d+1}^2 = r_0^2$, with $\epsilon = +1$ for the sphere, $\epsilon = -1$ for the hyperboloid (for a review, see [4]).

The potential of the oscillator has also been generalized for the complex projective space $\mathbb{C}P^N$, $N > 1$. That is defined as follows [3]:

$$V_{\text{osc}} = \omega^2 g^{\bar{a}b}\partial_{\bar{a}}K\partial_b K \tag{1.2}$$

where $K(z, \bar{z}) = \log(1 + z\bar{z})$ is the Kähler potential of $\mathbb{C}P^N$.

The generalized systems preserve the property of 'maximal superintegrability' of the conventional oscillator and Coulomb systems. They have two-dimensional $(2d - 1)$ functionally independent constants of motion (here d is the dimension of configuration space). The definition of the oscillator potential (1.2) tells us to define the Coulomb potential on $\mathbb{C}P^N$ as follows:

$$V_{\text{Coulomb}} = -\frac{\gamma}{\sqrt{g^{\bar{a}b}\partial_{\bar{a}}K\partial_b K}}. \tag{1.3}$$

In some cases one can establish the non-trivial relation between oscillator and Coulomb systems: the $(p+1)$-dimensional Coulomb problem can be obtained from the $2p$-dimensional oscillator by the so-called Levi-Civita (or Bohlin) $(p = 1)$, Kustaanheimo–Stiefel $(p = 2)$ and Hurwitz transformations $(p = 4)$, when $p = 1, 2, 4$ [5], corresponding to the reduction by the actions of Z_2, $U(1)$ and $SU(2)$ groups, respectively. To be more precise, these transformations connect the energy levels of oscillators with the one-parametric families of Coulomb-like systems, specified by the presence of a magnetic flux for $p = 1$ [6]; by a Dirac monopole for $p = 2$ (the MIC–Kepler system)[7]; and by a Yang monopole for $p = 4$ [8] (for a review, see [9]). Among these systems the most elegant (and important) one is, probably, the MIC–Kepler system, describing the relative motion of two Dirac dyons. It is also relevant to the scattering of two well-separated BPS monopoles and dyons. The latter system was considered in a well-known paper by Gibbons and Manton [10], where the existence of a hidden Coulomb-like symmetry was established. Nowadays the MIC–Kepler system is studied to no less extent than the Coulomb system [11].

To relate the four-dimensional oscillator with the MIC–Kepler system, we have to perform its Hamiltonian reduction by the action of the $U(1)$ group which leads the canonical symplectic structure on $T^*\mathbb{C}^2$ to the twisted symplectic structure on $T^*\mathbb{R}^3$ specified by the presence of a monopole magnetic field

$$\Omega_{\text{can}} = \mathrm{d}z \wedge \mathrm{d}\pi + \mathrm{d}\bar{z} \wedge \mathrm{d}\bar{\pi} \quad \to \Omega_{\text{red}} = \mathrm{d}\mathbf{x} \wedge \mathrm{d}\mathbf{p} + s\frac{\mathbf{x} \times \mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}}{|\mathbf{x}|^3}. \tag{1.4}$$

Here s denotes the value of the generator of the Hamiltonian action

$$s = \frac{J_0}{2} \qquad J_0 = \mathrm{i}(z\pi - \bar{z}\bar{\pi}). \tag{1.5}$$

The reduced coordinates are connected with the initial one as follows:

$$\mathbf{x} = z\boldsymbol{\sigma}\bar{z} \qquad \mathbf{p} = \frac{z\boldsymbol{\sigma}\pi + \bar{\pi}\boldsymbol{\sigma}\bar{z}}{2z\bar{z}} \tag{1.6}$$

where $\boldsymbol{\sigma}$ denote Pauli matrices.

Upon this reduction, the energy surface of the four-dimensional oscillator yields that of the MIC–Kepler system. Applying this reduction to the oscillator on a four-dimensional sphere/hyperboloid and on the complex projective space $\mathbb{C}P^2$ we shall get the MIC–Kepler system on a three-dimensional hyperboloid [3, 12].

The appearance, in the reduced system, of the monopole field is due to Hamiltonian reduction: that corresponds to the compactification of the spatial degree of freedom in the circle, which generates the magnetic charge. So, the above reduction could be used for the construction of the three-dimensional systems with the monopole from the four-dimensional systems. Vice versa, one can try to construct the superintegrable four-dimensional system (without monopole) by lifting the given three-dimensional superintegrable system.

In the present paper we analyse the following question: whether the maximally superintegrable systems on four-dimensional $U(2)$-invariant Kähler spaces, whose reductions yield the (superintegrable) three-dimensional oscillator and Coulomb systems with monopoles, including the systems on the configurational spaces with non-constant curvature, exist.

For this purpose we reduce the Hamiltonian system on the four-dimensional space equipped with $U(2)$-invariant Kähler metric to the system on three-dimensional conformal-flat space (see section 2). We find that the oscillator and Coulomb systems on the three-dimensional space, sphere and hyperboloid are originated on the four-dimensional oscillator and Coulomb systems on some Kähler spaces with conic singularity, so that (1.2) and (1.3) give us the well-defined generalizations of the oscillator and Coulomb potentials on Kähler spaces. However, in the presence of a Dirac monopole field that arises due to Hamiltonian reduction, the trajectories of the three-dimensional systems become unclosed. Hence, in general, these systems are not superintegrable. On the other hand, one can define the 'maximally superintegrable' generalization of the three-dimensional oscillator with a Dirac monopole, which originates in the four-dimensional system with non-Kähler metrics (see section 3). We also find the family of superintegrable four-dimensional oscillators, which yields the 'maximally superintegrable' oscillator with monopoles, on the three-dimensional spaces with non-constant curvature (see section 4).

2. Three-dimensional systems with monopoles from $U(2)$-invariant Kähler spaces

As we mentioned in the introduction, the Hamiltonian reduction, by the action of $U(1)$ group, of the eight-dimensional canonical symplectic structure yields the six-dimensional canonical symplectic structure twisted by the magnetic field of the Dirac monopole (1.4). A particular consequence of this reduction is the reduction of the energy surface of the oscillator (on $\mathbb{C}^2$, S^4, $\mathbb{C}P^2$) to the energy surface of the MIC–Kepler system (on $\mathbb{R}^3$ and AdS_3).

In this section we would like to reveal which sort of system arises upon $U(1)$ Hamiltonian reduction of the four-dimensional mechanical systems on the spaces with $U(2)$-invariant Kähler metrics. Particularly, we hope to find, in this way, the four-dimensional origins of the three-dimensional oscillator and Coulomb systems on Euclidean spaces, spheres and hyperboloids (which are superintegrable systems).

The Kähler potential of the $U(2)$-invariant Kähler spaces M, $\dim_{\mathbb{C}} M_0 = 2$ is (in the appropriate local coordinates $z^a, a = 1, 2$) of the form $K(z\bar{z})$. Hence, the corresponding metric reads

$$g_{a\bar{b}} = \frac{\partial^2 K(z\bar{z})}{\partial z^a \partial \bar{z}^b} = a\delta_{a\bar{b}} + a'\bar{z}^a z^b \qquad \text{where} \quad a = \mathrm{d}K'(y)/\mathrm{d}y \quad a' = a'(y). \tag{2.1}$$

The particular cases of these spaces are the Euclidean space $\mathbb{C}^2$ (when $K = z\bar{z}$) and the complex projective space $\mathbb{C}P^2$ (when $K = \log(1 + z\bar{z})$).

The motion of the particle on M in the $U(2)$-invariant potential field V is described by the following Hamiltonian system:

$$\Omega_{\text{can}} = \mathrm{d}z^a \wedge \mathrm{d}\pi_a + \mathrm{d}\bar{z}^a \wedge \mathrm{d}\bar{\pi}_a \qquad \mathcal{H} = g^{a\bar{b}}\pi_a\bar{\pi}_b + V(z\bar{z}). \tag{2.2}$$

The Noether constants of motion corresponding to $U(2)$ symmetry are given by the generators

$$\mathbf{J} = \mathrm{i}z\boldsymbol{\sigma}\pi - \mathrm{i}\bar{\pi}\boldsymbol{\sigma}\bar{z} \qquad J_0 = \mathrm{i}z\pi - \mathrm{i}\bar{z}\bar{\pi} : \quad \{J_0, J_k\} = 0 \qquad \{J_k, J_l\} = 2\epsilon_{klm}J_m \tag{2.3}$$

where $\boldsymbol{\sigma}$ denotes standard Pauli matrices.

In order to perform the Hamiltonian reduction of this system by the action of the generator J_0, we have to fix its level surface,

$$J_0 = 2s \tag{2.4}$$

and then factorize the level surface by the action of vector field $\{J_0, \ \}$. The resulting six-dimensional phase space $T^*M^{\rm red}$ could be parametrized by the following $U(1)$-invariant functions:

$$\mathbf{y} = z\boldsymbol{\sigma}\bar{z} \qquad \boldsymbol{\pi} = \frac{z\boldsymbol{\sigma}\pi + \bar{\pi}\boldsymbol{\sigma}\bar{z}}{2z\bar{z}} : \quad \{\mathbf{y}, J_0\} = \{\boldsymbol{\pi}, J_0\} = 0. \tag{2.5}$$

In these coordinates the reduced symplectic structure and the generators of the angular momentum are given by the expressions

$$\Omega_{\rm red} = \mathrm{d}\boldsymbol{\pi} \wedge \mathrm{d}\mathbf{y} + s\frac{\mathbf{y} \times \mathrm{d}\mathbf{y} \times \mathrm{d}\mathbf{y}}{|\mathbf{y}|^3} \qquad \mathbf{J}_{\rm red} = \mathbf{J}/2 = \boldsymbol{\pi} \times \mathbf{y} + s\frac{\mathbf{y}}{|\mathbf{y}|}. \tag{2.6}$$

The reduced Hamiltonian is given by the expression

$$\mathcal{H}_{\rm red} = \frac{1}{a}[y\boldsymbol{\pi}^2 - b(\mathbf{y}\boldsymbol{\pi})^2] + s^2\frac{1-by}{ay} + V(y) \qquad \text{where} \quad y \equiv |\mathbf{y}| \qquad b = \frac{a'(y)}{a + ya'(y)}. \tag{2.7}$$

Hence, the reduced system is specified by the presence of a Dirac monopole.

Let us perform the canonical transformation $(\mathbf{y}, \boldsymbol{\pi}) \to (\mathbf{x}, \mathbf{p})$ to the coordinates, where the metric takes a conformal-flat form:

$$\mathbf{x} = f(y)\mathbf{y} \qquad \pi = f\mathbf{p} + \frac{\mathrm{d}f}{\mathrm{d}y}\frac{(\mathbf{y}\mathbf{p})}{y}\mathbf{y} \tag{2.8}$$

where

$$\left(1 + \frac{yf'(y)}{f}\right)^2 = 1 + \frac{ya'(y)}{a} \quad \Rightarrow \quad \left(\frac{\mathrm{d}\log x}{\mathrm{d}y}\right)^2 = \frac{\mathrm{d}\log ya(y)}{y\,\mathrm{d}y}. \tag{2.9}$$

Note, that $x < 1$.

In the new coordinates the Hamiltonian takes the form:

$$\mathcal{H}_{\rm red} = \frac{x^2(y)}{ya(y)}\mathbf{p}^2 + \frac{s^2}{y(a + ya'(y))} + V(y(x)). \tag{2.10}$$

In order to express $y, a(y), a'(y)$ via x it is convenient to introduce the function

$$\tilde{A}(y) \equiv \int (a + ya'(y))yf(y)\,\mathrm{d}y \tag{2.11}$$

and consider its Legendre transform $A(x)$,

$$A(x) = xa(y)y - \tilde{A}(y). \tag{2.12}$$

Then, we get immediately

$$\frac{\mathrm{d}A(x)}{\mathrm{d}x} = a(y)y \qquad x\frac{\mathrm{d}^2A}{\mathrm{d}x^2} = y\sqrt{a(a + ya'(y))}. \tag{2.13}$$

By using these expressions, we can present the reduced Hamiltonian system as follows:

$$\mathcal{H}_{\rm red} = \frac{x^2}{N^2}\mathbf{p}^2 + \frac{s^2}{(2xN'(x))^2} + V(y(x)) \qquad \Omega_{\rm red} = \mathrm{d}\mathbf{p} \wedge \mathrm{d}\mathbf{x} + s\frac{\mathbf{x} \times \mathrm{d}\mathbf{x} \times \mathrm{d}\mathbf{x}}{|\mathbf{x}|^3} \tag{2.14}$$

where

$$N^2(x) \equiv \frac{\mathrm{d}A}{\mathrm{d}x}. \tag{2.15}$$

The Kähler potential of the initial system is connected with N via equations

$$\frac{\mathrm{d}K}{\mathrm{d}x} = \frac{N^3(x)}{2x^2 N'(x)} \qquad \frac{\mathrm{d}\log y}{\mathrm{d}x} = \frac{N}{2x^2 N'(x)}. \tag{2.16}$$

Let us postulate that the 'oscillator potential' on the spaces under consideration acts by the same formula as on the complex projective spaces (1.2). Then, upon reduction it will read as follows:

$$V_{\mathrm{osc}} = \omega^2 \partial_{\bar{a}} K g^{\bar{a}b} \partial_b K = \omega^2 \frac{y a^2(y)}{a(y) + y a'(y)} = \left(\omega \frac{N^2}{2xN'(x)}\right)^2. \tag{2.17}$$

Similarly, we could choose the 'Coulomb potential' (1.3), and get its reduced version

$$V_{\mathrm{Coulomb}} = -\gamma \frac{2xN'(x)}{N^2(x)}. \tag{2.18}$$

In further studies we will need to consider the classical trajectories of the (reduced) system, in order to check their closedness (closedness of trajectories is the explicit indication of superintegrability).

For this purpose it is convenient to direct the x_3 axis along $\mathbf{J}$, i.e. assume that $J = J_3$. Upon this choice of coordinate system one has

$$\frac{x_3}{x} = \frac{s}{J}. \tag{2.19}$$

Then, after obvious manipulations, we get

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{2J}{N^2} \qquad \mathcal{E} = \frac{J^2 - s^2}{N^2} + \frac{J^2}{x^2 N^2}\left(\frac{\mathrm{d}x}{\mathrm{d}\phi}\right)^2 + \frac{s^2}{(2xN'(x))^2} + V(x) \tag{2.20}$$

where

$$\phi = \arctan \frac{x_1}{x_2}. \tag{2.21}$$

From the expression (2.20) we find,

$$\left|\frac{\phi}{J}\right| = \int \frac{\mathrm{d}x}{x\sqrt{(\mathcal{E} - V_{\mathrm{eff}})N^2 - J^2 + s^2}} \qquad \text{where} \quad V_{\mathrm{eff}} = V(x) + \frac{s^2}{(2x\,\mathrm{d}N/\mathrm{d}x)^2}. \tag{2.22}$$

2.1. Euclidean space

Let us consider the simplest case, when the reduced configuration space is $\mathbb{R}^3$, i.e. $N = \sqrt{2}x$. In this case the reduced Hamiltonian reads

$$\mathcal{H}_{\mathrm{red}} = \frac{\mathbf{p}^2}{2} + \frac{s^2}{8x^2} + V(x). \tag{2.23}$$

The trajectories of the system are defined by equations (2.19) and

$$\left|\frac{\phi}{J}\right| = \int \frac{\mathrm{d}x}{x\sqrt{2(\mathcal{E} - V)x^2 - J^2 + 3s^2/4}} \tag{2.24}$$

The Kähler potential and metric of the original four-dimensional system are of the form:

$$K = (z\bar{z})^4 \qquad g_{a\bar{b}} = 4(z\bar{z})^2[(z\bar{z})\delta_{a\bar{b}} + 3\bar{z}^a z^b]. \tag{2.25}$$

Hence, the systems on $\mathbb{R}^3$ are originated on the Kähler conifold[3].

[3] We thank Dmitry Fursaev for this remark.

Note that the oscillator and Coulomb potentials (1.2) and (1.3) take, on this conifold, the following form:

$$V_{\rm Coulomb} = -\frac{\gamma}{(z\bar{z})^2} \qquad V_{\rm osc} = \omega^2 (z\bar{z})^4. \tag{2.26}$$

Upon reduction they yield the oscillator and Coulomb potentials on $\mathbb{R}^3$!

On the other hand, for the

$$V_{\rm eff} = \frac{s^2}{x^2} + V(x)$$

one has

$$\left|\frac{\phi}{J}\right| = \int \frac{{\rm d}x}{x\sqrt{2(\mathcal{E} - V)x^2 - J^2}} \tag{2.27}$$

so that the form of trajectory, $\phi(x)$, is independent of the 'monopole number' s.

Hence, the well-defined monopole generalization of the system on $\mathbb{R}^3$ with potential $V(x)$ reads

$$\mathcal{H}_s = \frac{\mathbf{p}^2}{2} + \frac{s^2}{2x^2} + V(x) \qquad \Omega_{\rm red} = {\rm d}\mathbf{p} \wedge {\rm d}\mathbf{x} + s\frac{\mathbf{x} \times {\rm d}\mathbf{x} \times {\rm d}\mathbf{x}}{|\mathbf{x}|^3}. \tag{2.28}$$

Its four-dimensional origin is formulated as follows:

$$\Omega_{\rm can} = {\rm d}z^a \wedge {\rm d}\pi_a + {\rm d}\bar{z}^a \wedge {\rm d}\bar{\pi}_a \qquad \mathcal{H} = g^{\bar{a}b}\bar{\pi}_a \pi_b + \frac{3J_0^2}{16(z\bar{z})^4} + V(z\bar{z}) \tag{2.29}$$

where $g_{\bar{a}b}$ is given by (2.25).

3. Sphere and hyperboloid

In this section we consider the particular case of our construction, when the reduced system (2.14) is formulated on the three-dimensional sphere or (two-sheet) hyperboloid.

For this purpose we choose the following value of N:

$$N = 2\sqrt{2}r_0 x/(1 + \epsilon x^2) \qquad \epsilon = 1, -1. \tag{3.1}$$

Here $\epsilon = 1$ corresponds to the sphere, $\epsilon = -1$ corresponds to the two-sheet hyperboloid.

The corresponding Hamiltonian is of the form:

$$\mathcal{H}_{\rm red} = \frac{(1 + \epsilon x^2)^2}{8r_0^2}\left(\mathbf{p}^2 + \frac{s^2}{4x^2}\right) + V(x) + \frac{s^2 x^2}{2r_0^2(1 - \epsilon x^2)^2} + \frac{\epsilon s^2}{8r_0^2}. \tag{3.2}$$

Solving equations (2.16) we could find the Kähler space, where system (2.14) originates. It is defined by the following Kähler potential and metric:

$$K = \frac{\epsilon r_0^2}{2}\log(1 + 4\epsilon(z\bar{z})^4) \qquad g_{a\bar{b}} = \frac{8r_0^2(z\bar{z})^2}{1 + 4\epsilon(z\bar{z})^4}\left[(z\bar{z})\delta_{a\bar{b}} + \frac{3 - 4\epsilon(z\bar{z})^4}{(1 + 4\epsilon(z\bar{z})^4)}\bar{z}^a z^b\right]. \tag{3.3}$$

Hence, the systems on the sphere and two-sheet hyperboloid also originate on the Kähler conifolds.

On these conifolds the oscillator and Coulomb potentials (1.2) and (1.3) read as follows:

$$V_{\rm Coulomb} = -\frac{\gamma}{\sqrt{2}r_0(z\bar{z})^2} \qquad V_{\rm osc} = 2\omega^2 r_0^2 (z\bar{z})^4. \tag{3.4}$$

Upon reduction to the sphere and hyperboloid they take the form:

$$V_{\rm Coulomb} = -\frac{\sqrt{2}\gamma}{r_0}\frac{1 - \epsilon x^2}{2x} \qquad V_{\rm osc} = \omega^2 r_0^2 \frac{2x^2}{(1 - \epsilon x^2)^2}. \tag{3.5}$$

These potentials are precisely Coulomb and oscillator potentials on the sphere and hyperboloid (1.1) written in conformal-flat coordinates. Hence, the oscillator/Coulomb system on the conifold (3.3) reduces, for $J_0 = 0$, to the oscillator/Coulomb system on the three-dimensional sphere/hyperboloid. Hence, the initial four-dimensional oscillator and Coulomb systems are superintegrable systems, when the constant of motion J_0 takes the value $J_0 = 0$.

When $J_0 \neq 0$, the relation between three- and four-dimensional systems is more complicated, and needs separate consideration of the oscillator and Coulomb cases.

Let us consider, first, the case of an oscillator. To check the superintegrability, let us clarify whether the trajectories of the reduced system are closed.

Substituting (3.1) and (3.5) in (2.22), we get

$$\left|\frac{\phi}{J}\right| = \int \frac{\mathrm{d}u}{\sqrt{-4r_0^2\left(\omega^2 r_0^2 + 2\mathcal{E}\right) + \left(8r_0^2\mathcal{E} + l^2\right)u - (s^2 + l^2)u^2}} \tag{3.6}$$

where

$$l^2 = 4(J^2 - s^2) \qquad 4u = (x + 1/x)^2. \tag{3.7}$$

From this expression we easily get

$$\left(x + \frac{1}{x}\right)^2 = 8\frac{2r_0^2\mathcal{E} + J^2 - s^2}{4J^2 - 3s^2}\left(1 + \sqrt{1 - 4\frac{\left(2r_0^2\mathcal{E} + r_0^4\omega^2\right)(4J^2 - 3s^2)}{\left(2r_0^2\mathcal{E} + J^2 - s^2\right)^2}}\sin 2\sqrt{1 - \frac{3s^2}{4J^2}}|\phi|\right). \tag{3.8}$$

Hence, trajectories are closed only when

$$\sqrt{1 - \frac{3s^2}{4J^2}} \quad \text{is a rational } \textit{number}.$$

Particularly, trajectories are closed in the 'ground state', i.e. for $s = J$. In this case they belong to the 'equatorial plane', $x_3 = x$. Hence, the Hamiltonian system (2.14) on a sphere/hyperboloid with the oscillator potential is not superintegrable for arbitrary value of monopole number s.

However, one can get the monopole generalization of an oscillator whose trajectories are closed for any s, choosing the potential

$$V_{\mathrm{osc}}^s = V_{\mathrm{osc}} + \frac{3s^2}{4x^2(\mathrm{d}N/\mathrm{d}x)^2}. \tag{3.9}$$

In this case the trajectories are given by the expression

$$\left(x + \frac{1}{x}\right)^2 = 2\frac{2r_0^2\mathcal{E} + J^2 - s^2}{2J^2}\left(1 + \sqrt{1 - 16J^2 r_0^2\frac{2\mathcal{E} + \omega^2 r_0^2}{\left(2r_0^2\mathcal{E} + J^2 - s^2\right)^2}}\sin 2|\phi|\right) \tag{3.10}$$

i.e. they are closed for any s.

Hence, the superintegrable generalization of the Higgs oscillator specified by the presence of a Dirac monopole is defined by the Hamiltonian

$$\mathcal{H}_{\mathrm{MIC-osc}}^{\epsilon} = \frac{(1 + \epsilon x^2)^2}{8r_0^2}\left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) + \left(\omega^2 r_0^2 + \frac{s^2}{4r_0^2}\right)\frac{2x^2}{(1 - \epsilon x^2)^2} \qquad \epsilon = \pm 1 \tag{3.11}$$

where $\epsilon = 1$ corresponds to the sphere and $\epsilon = -1$ to the hyperboloid.

It originates in the Hamiltonian given by the expression

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{3J_0^2}{16R(z\bar{z})} + \omega^2 K_a g^{a\bar{b}} K_{\bar{b}} \qquad R = \frac{32r_0^2(z\bar{z})^4}{(1 + 4\epsilon(z\bar{z})^4)^2}. \tag{3.12}$$

There is an important difference of the above reduced oscillator from that on the $\mathbb{R}^3$.

Namely, the four-dimensional system with 'frequency' ω yields the three-dimensional oscillator with 'frequency' dependent on the 'monopole number' s

$$\omega_s = \sqrt{\omega^2 + \frac{s^2}{4r_0^4}}$$

while the frequency of the oscillator reduced to $\mathbb{R}^3$ is independent of s.

Now, let us consider the system with Coulomb potential (1.3) on the conifold with Kähler structure (3.3). After Hamiltonian reduction it yields the three-dimensional system with Hamiltonian (3.2) where $V = V_{\text{Coulomb}}$ (3.5).

On the level surface $s = 0$ (i.e. in the absence of a Dirac monopole), the reduced system coincides with the standard Coulomb system on the sphere/hyperboloid. Therefore it is superintegrable. On the level surface $s \neq 0$ (i.e. in the presence of a monopole) the potential of the reduced system is the superposition of the Coulomb potential and that of the oscillator, proportional to s^2/r_0^4! So, it is not surprising that the expression for trajectories of the reduced system $\phi = \phi(r)$ is given by the elliptic integral....

On the other hand, there are superintegrable MIC–Kepler systems on the sphere and hyperboloid, given by the Hamiltonian

$$\mathcal{H}^{\epsilon}_{\text{MIC}} = \frac{(1+\epsilon x^2)^2}{8r_0^2}\left(\mathbf{p}^2 + \frac{s^2}{x^2}\right) - \frac{\gamma}{r_0}\frac{1-\epsilon x^2}{2x} \qquad \epsilon = \pm 1 \tag{3.13}$$

where $\epsilon = 1$ corresponds to the sphere [13] and $\epsilon = -1$ to the hyperboloid [12].

To recover this system, we can try to modify the initial Coulomb system, transiting to the non-Kähler metric, as in the case of an oscillator (compare with (3.12)),

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{3J_0^2}{16R(z\bar{z})} + \frac{\gamma}{\sqrt{K_a g^{a\bar{b}} K_{\bar{b}}}} \qquad R = \frac{32r_0^2(z\bar{z})^4}{(1+\epsilon(z\bar{z})^4)^2}. \tag{3.14}$$

However, in that case the reduced Hamiltonian is given by that of the MIC–Kepler system (3.11) with additional oscillator potential. So even the modified Coulomb system is not exactly solvable for any value of J_0.

We have found the four-dimensional oscillator and Coulomb systems on appropriate Kähler conifolds, which result, after Hamiltonian reduction, in the oscillator and Coulomb systems on a three-dimensional sphere and a two-sheet hyperboloid. The appearance, in the reduced system, of the Dirac monopole, breaks the superintegrability of the system. However, the superintegrability of the oscillator system, as opposed to the Coulomb system, could be restored by the transition to the non-Kähler metric.

4. Family of oscillator systems

The results of the previous section could easily be extended to the Kähler space whose metric is defined by the potential

$$K = \frac{\epsilon r_0^2}{2}\log(1 + 4\epsilon(z\bar{z})^n) \qquad \epsilon = \pm 1 \quad n > 0. \tag{4.1}$$

For $n = 1$ the potential (4.1) defines the Fubini–Studi metric of the two-dimensional complex projective space $\mathbb{C}\mathrm{P}^2$ (for $\epsilon = 1$) and its non-compact version, the four-dimensional Lobachewski space $\mathcal{L}_2$ (for $\epsilon = -1$). These spaces are of constant curvature, and have the isometry group $SU(3)$ for $\epsilon = 1$ and $SU(1.2)$ for $\epsilon = -1$.

The case $n = 4$ was considered in the previous section. The system on such spaces results, after Hamiltonian reduction, in those on the sphere ($\epsilon = 1$) or two-sheet hyperboloid

$\epsilon = -1$ (which have constant curvature, and the isometry groups $SO(4)$ and $SO(1.3)$, respectively). For any other n both initial and reduced spaces have non-constant curvature and conic singularity.

The Hamiltonian systems on the spaces with Kähler potential (4.1) result, after reduction, in three-dimensional systems (2.14), with

$$N^2 = 2nr_0^2 \frac{x^{\sqrt{n}}}{\left(1+\epsilon x^{\sqrt{n}}\right)^2}. \tag{4.2}$$

Hence, the metric of the reduced configuration space is given by the expression

$$\mathrm{d}s^2 = \frac{2nr_0^2 x^{\sqrt{n}-2}(\mathrm{d}\mathbf{x})^2}{\left(1+\epsilon x^{\sqrt{n}}\right)^2} \tag{4.3}$$

so that for $n \neq 4$ it has a conifold structure[4].

The oscillator potential (2.17) is as follows:

$$V_{\rm osc} = 2r_0^2\omega^2(z\bar{z})^n. \tag{4.4}$$

It reduces to the following form:

$$V_{\rm osc}^{\rm red} = 2r_0^2\omega^2 \frac{x^{\sqrt{n}}}{\left(1-\epsilon x^{\sqrt{n}}\right)^2}. \tag{4.5}$$

The trajectories of the reduced oscillator are given by the expression

$$\left|\frac{\phi}{J}\right| = \int \frac{\mathrm{d}u}{\sqrt{-nr_0^2\left(r_0^2\omega^2+2\mathcal{E}\right) + \left(2nr_0^2\mathcal{E}+l^2\right)u - \left(\frac{4}{n}s^2+l^2\right)u^2}} \tag{4.6}$$

where

$$l^2 = 4(J^2-s^2) \qquad 4u = \left(x^{\sqrt{n}/2} + 1/x^{\sqrt{n}/2}\right)^2. \tag{4.7}$$

From this expression we easily get

$$\left(x^{\sqrt{n}/2} + x^{-\sqrt{n}/2}\right)^2 = \frac{nr_0^2\mathcal{E} + 2(J^2-s^2)}{2(J^2-s^2(1-1/n))} \times \left(1 + \sqrt{1 - 4nr_0^2 \frac{\left(2\mathcal{E}+r_0^2\omega^2\right)(J^2-s^2(1-1/n))}{\left(\mathcal{E}r_0^2+(J^2-s^2)\right)^2}}\, \sin 2\sqrt{1-\frac{(n-1)s^2}{nJ^2}}|\phi|\right). \tag{4.8}$$

Hence, trajectories are closed when the following condition holds:

$$\sqrt{1-\frac{(n-1)s^2}{nJ^2}} \quad \text{is a rational } number.$$

Therefore, trajectories are closed for any s only when $n = 1$, i.e. on the complex projective space $\mathbb{CP}^2$ (for $\epsilon = 1$) and on its non-compact version, four-dimensional Lobachewski space $\mathcal{L}^2 = SU(1.2)/U(1) \times SU(2)$. In this case the potential takes a quite simple form, $V = 2\omega^2 r_0^2 z\bar{z}$. The closedness of trajectories are due to the hidden symmetries of the system, given by the expressions [3]

$$\mathbf{I} = \frac{J_+\boldsymbol{\sigma}J_-}{2r_0^2} + 2r_0^2\omega^2 z\boldsymbol{\sigma}\bar{z} \qquad J_a^+ = \pi_a + \epsilon(\bar{\pi}\bar{z})\bar{z}^a \qquad J^- = \bar{J}^+ \tag{4.9}$$

[4] In the vicinity of a singularity this metric could be presented as follows: $\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\bar{\Omega}^2$, where $R = r^{\sqrt{n}/2}/\sqrt{n}/2$, $\mathrm{d}\bar{\Omega}^2 = (n/2)^2\,\mathrm{d}\Omega^2$ where $\mathrm{d}\Omega^2$ is a metric on S^2. Hence, the solid angle around the singularity is equal to $n\pi$, instead of 4π (D Fursaev).

where $J_a^\pm$ are the translation generators. The reduced Hamiltonian is of the form:

$$\mathcal{H}_{\mathrm{red}} = \frac{x(1+\epsilon x)^2\mathbf{p}^2}{2r_0^2} + s^2\frac{(1+\epsilon x)^4}{2r_0^2 x(1-\epsilon x)^2} + \frac{2r_0^2\omega^2 x}{(1-\epsilon x)^2}. \tag{4.10}$$

Fixing the energy surface $\mathcal{H} = E_{\mathrm{osc}}$ of the reduced system, we can transform it into the MIC–Kepler system on a hyperboloid, given by the Hamiltonian (3.11).

For $n \neq 1$, we could get the superintegrable oscillator with monopole, choosing the potential

$$V_{\mathrm{osc}}^s = V_{\mathrm{osc}} + \frac{(n-1)}{n}\frac{s^2}{x^2(\mathrm{d}N/\mathrm{d}x)^2}. \tag{4.11}$$

In this case the Hamiltonian of the reduced system reads

$$\mathcal{H} = \frac{\left(1+\epsilon x^{\sqrt{n}}\right)^2}{2nr_0^2 x^{\sqrt{n}-2}}\mathbf{p}^2 + s^2\frac{\left(1+\epsilon x^{\sqrt{n}}\right)^4}{2nr_0^2 x^{\sqrt{n}}\left(1-\epsilon x^{\sqrt{n}}\right)^2} + 2r_0^2\omega^2\frac{x^{\sqrt{n}}}{\left(1-\epsilon x^{\sqrt{n}}\right)^2} \tag{4.12}$$

while the trajectories are given by equation

$$\left(x^{\sqrt{n}/2} + x^{-\sqrt{n}/2}\right)^2 = \frac{nr_0^2\mathcal{E} + 2(J^2 - s^2)}{J^2}\left(1 + \sqrt{1 - 4nr_0^2\frac{(2\mathcal{E} + r_0^2\omega^2)J^2}{nr_0^2\mathcal{E} + 2(J^2-s^2)^2}}\sin 2|\phi|\right). \tag{4.13}$$

This superintegrable oscillator with monopole originates in the four-dimensional system with Hamiltonian

$$\mathcal{H} = g^{\bar{a}b}\bar{\pi}_a\pi_b + \frac{(n-1)J_0^2}{4nR(z\bar{z})} + 2r_0^2\omega^2(z\bar{z})^n \qquad R = \frac{2n^2r_0^2(z\bar{z})^4}{(1+4\epsilon(z\bar{z})^n)^2} \tag{4.14}$$

where $g^{\bar{a}b}$ is defined by the Kähler potential (4.1).

5. Conclusion

We considered the reduction of the mechanical systems on four-dimensional Kähler spaces with $U(2)$ isometry to the three-dimensional systems, paying special attention to the 'oscillator' and 'Coulomb' systems, defining their potentials by expressions (1.2) and (1.3), respectively. From a previous study [3] it was known that such an 'oscillator' potential defines the well-defined superintegrable system on $\mathbb{C}\mathrm{P}^n$, and is distinguished with respect to supersymmetrization as well. Since the Hamiltonian reduction by the action of the $U(1)$ group generates, in the resulting three-dimensional system, the magnetic field of a Dirac monopole, we hope to find, in this way, the superintegrable generalizations of oscillator and Coulomb systems on curved spaces, specified by the presence of a Dirac monopole. Particularly, we found the four-dimensional Kähler spaces (with conic singularities) where the three-dimensional oscillator and Coulomb systems on the $\mathbb{R}^3$, S^3, $\mathbb{H}^3$ originate, and establish that the original oscillator and Coulomb potentials are, indeed, given by (1.2) and (1.3). However, when these four-dimensional systems result in the three-dimensional system specified by the presence of Dirac monopoles, their trajectories become unclosed. In other words, the monopole field breaks superintegrability of the system. In the case of an oscillator, transiting from the Kähler metric to the appropriate non-Kähler one, we can restore superintegrability of the systems (both the initial and the reduced one), but we cannot do the same for the Coulomb system. We also extended these considerations for some parametric family of Käher spaces including the previous ones as a particular case. We found that the unique representative of this family, where the oscillator is superintegrable, is the complex projective space $\mathbb{C}\mathrm{P}^2$ (and its non-compact version, Lobachewski space $\mathcal{L}_2 = SU(2.1)/SU(2) \times U(1)$. The energy surface of the oscillator on this four-dimensional space leads to the energy surface of the MIC–Kepler system on a three-dimensional hyperboloid. On the other hand, the Kähler configuration space is distinguished from the viewpoint of supersymmetry. Particularly, the above-considered systems could be supersymmetrized precisely as the oscillator on $\mathbb{C}P^N$ [3]. So, the requirement of superintegrability yields the breaking of this supersymmetry in the presence of the monopole field.

Finally, let us note that while previous superintegrable generalizations of oscillator systems were formulated on constant curvature spaces, the superintegrable oscillators constructed in the present paper could have configuration spaces with non-constant curvature and conic singularities.

Acknowledgments

We thank Dmitry Fursaev for the useful conversations, and Levon Mardoyan for the interest in the work. The work of AN is partially supported by grants INTAS 00-00262 and ANSEF PS81.

References

[1] Perelomov A M 1990 *Integrable Systems of Classical Mechanics and Lie Algebras* (Basle: Birkhauser)
[2] Higgs P W 1979 *J. Phys. A: Math. Gen.* **12** 309
Leemon H I 1979 *J. Phys. A: Math. Gen.* **12** 489
Schrödinger E 1941 *Proc. R. Irish Soc.* **46** 9
Schrödinger E 1941 *Proc. R. Irish Soc.* **46** 183
Schrödinger E 1941 *Proc. R. Irish Soc.* **47** 53
[3] Bellucci S and Nersessian A 2003 *Phys. Rev.* D **67** 065013
[4] Grosche C, Pogosyan G S and Sissakian A N 1996 *Phys. Part. Nucl.* **27** 244
Grosche C, Pogosyan G S and Sissakian A N 1997 *Ann. Phys., Lpz.* **6** 144
[5] Levi-Civita T 1906 *Op. Math.* **2** 411
Kustaanheimo P and Stiefel E 1965 *J. Reine Angew. Math.* **218** 204
Davtyan L S, Mardoyan L G, Pogosyan G S, Sissakian A N and Ter-Antonyan V M 1987 *J. Phys. A: Math. Gen.* **20** 6121
[6] Nersessian A, Ter-Antonian V and Tsulaia M M 1996 *Mod. Phys. Lett.* A **11** 1605
[7] Zwanziger D 1968 *Phys. Rev.* **176** 1480
McIntosh H V and Cisneros A 1970 *J. Math. Phys.* **11** 896
[8] Iwai T 1990 *J. Geom. Phys.* **7** 507
Mardoyan L G, Sissakian A N and Ter-Antonyan V M 1998 *Phys. At. Nucl.* **61** 1746
[9] Ter-Antonyan V M 2003 Dyon–oscillator duality *Preprint* quant-ph/0003106
[10] Gibbons G W and Manton N S 1986 *Nucl. Phys.* B **274** 183
See also Feher L G and Horvathy P A 1987 *Phys. Lett.* B **182** 183
[11] Iwai T and Uwano Y 1986 *J. Math. Phys.* **27** 1523
Mladenov I and Tsanov V 1987 *J. Phys. A: Math. Gen.* **20** 5865
Nersessian A and Ter-Antonyan V 1994 *Mod. Phys. Lett.* A **9** 2431
Nersessian A and Ter-Antonyan V 1995 *Mod. Phys. Lett.* A **10** 2633
Mardoyan L G, Sissakian A N and Ter-Antonyan V M 1997 *Int. J. Mod. Phys.* A **12** 237
Mardoyan L G, Petrosyan L S and Sarkisyan H 2003 *Phys. Rev.* A **68** 014103
Mardoyan L, Nersessian A and Petrosyan M 2003 *Preprint* quant-ph/0305027
[12] Nersessian A and Pogosyan G 2001 *Phys. Rev.* A **63** 020103(R)
[13] Gritsev V V, Kurochkin Yu A and Otchik V S 2000 *J. Phys. A: Math. Gen.* **33** 4903